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Supersymmetric Polychronakos Spin Chain: Motif, Distribution Function, and Character.

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ABSTRACT

Degeneracy patterns and hyper-multiplet structure in the spectrum of the su$(m|n)$ supersymmetric Polychronakos spin chain are studied by use of the "motif". Using the recursion relation of the supersymmetric Rogers–Szegö polynomials which are closely related to the partition function of the N spin chain, we give the representation for motif in terms of the supersymmetric skew Young diagrams. We also study the distribution function for quasi-particles. The character formulae for $N \to \infty$ are briefly discussed.

Key Words: integrable model, supersymmetry, Haldane–Shastry spin chain, Yangian, Young diagram, character formula

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1 Introduction

The integrable spin chain with an inverse square interaction has received extensive studies in these years. One of the famous model is the Haldane–Shastry (HS) spin chain [1,2]. The su(m) HS spin chain is integrable, and contrary to the ordinary Heisenberg spin chain it has the $\mathcal{Y}(s\ell_m)$ Yangian symmetry even in a finite chain [3]. While the quasi-particle (spinon) for the Haldane–Shastry has a square dispersion relation, there exists another integrable spin chain whose quasi-particle has a linear dispersion relation. This model, which we call the Polychronakos spin chain, was originally introduced in Ref. 4, and thanks to a linear dispersion relation we can compute the partition function [5, 6]. What is remarkable is that, similar to the case of HS model, the Polychronakos spin chain also possesses the $\mathcal{Y}(s\ell_m)$ Yangian symmetry [7].

From the viewpoint of the conformal field theory, it was realized [8,9] that the Yangian symmetry can be embedded into the level-1 WZNW theory, and that the first Yangian invariant operator is the Virasoro generator L_0. Indeed, this supports the fact that the energy spectrum for the Polychronakos spin chain is equally spaced. As the Polychronakos spin chain has the exact Yangian symmetry in the case of finite N spins, the character for the level-1 WZNW theory can be given in a large N limit of the partition function of the Polychronakos spin chain [7]. In this sense, the partition function of spin chain with finite number of lattice sites is viewed as the *restricted* character formula. In a computation of such partition function, the Yangian invariant bases which are called the "motifs" [3, 10] play an important role. These motifs span the Fock space of the Yangian invariant system, and the degeneracy for each motif can be given from the representation theory of the Yangian algebra. The representation for the motif was given for su(2) case in the original paper [3], and for su(m) case it was established by one of the author [7, 11], where the main observation is that the partition function for finite-site Polychronakos spin chain can be identified with the Rogers–Szegö (RS) polynomial [12] at some special values.

In this paper we consider the su($m|n$) supersymmetric extension of the Polychronakos

spin chain (SP model), whose Hamiltonian is written as

$$\mathcal{H}^{(m|n)} = \sum_{1\le i<j\le N} \frac{1-P_{ij}}{(z_i-z_j)^2}. \tag{1.1}$$

Here z_i (for $i=1,2,\ldots,N$) are zeros of the N-th Hermite polynomial. The supersymmetric spin operator P_{ij} is written as [10]

$$P_{ij} = \sum_{\alpha,\beta=1}^{n+m} c_{i,\alpha}^{\dagger}\, c_{j,\beta}^{\dagger}\, c_{i,\beta}\, c_{j,\alpha}, \tag{1.2}$$

Here the creation–annihilation operators, $c_{i,\alpha}^{\dagger}$ and $c_{i,\alpha}$, are

$$c_{i,\alpha} : \begin{cases} \text{bosonic} & \text{for } \alpha = 1,\ldots,m, \\ \text{fermionic} & \text{for } \alpha = m+1,\ldots,m+n\ , \end{cases}$$

and we have a constraint

$$\sum_{\alpha=1}^{m+n} c_{i,\alpha}^{\dagger}\, c_{i,\alpha} = 1.$$

Note that the spin operator P_{ij} is a permutation operator, satisfying

$$P_{ij}\,P_{jk} = P_{jk}\,P_{ki} = P_{ki}\,P_{ij}, \qquad\qquad {P_{ij}}^2 = 1.$$

As was shown in Ref. 6, the su(m) Polychronakos spin chain is a static limit of the spin Calogero model confined in the harmonic potential. We can then compute the partition function of the su(m) Polychronakos spin model by factoring out the dynamical degree of freedom from the spin Calogero model. By using a similar approach for the case of SP model (1.1), one can obtain the corresponding partition function $\mathcal{Z}_N^{(m|n)}(q) = \operatorname{Tr} q^{\mathcal{H}^{(m|n)}}$ as [13]

$$\mathcal{Z}_N^{(m|n)}(q) = \sum_{\substack{\sum_{i=1}^m a_i + \sum_{j=1}^n b_j = N \\ a_i\ge 0,\quad b_j\ge 0}} \frac{(q;q)_N}{\prod_{i=1}^m (q;q)_{a_i} \cdot \prod_{j=1}^n (q;q)_{b_j}} \cdot q^{\frac{1}{2}\sum_{j=1}^n b_j\,(b_j-1)} . \tag{1.3}$$

See § 2 for definitions. We note that the partition function (1.3) has the duality for $m, n \neq 0$,

$$\mathcal{Z}_N^{(m|n)}(q) = q^{\frac{N(N-1)}{2}} \mathcal{Z}_N^{(n|m)}(q^{-1}).$$

We remark that in the pure fermion case, *i.e.*, when $m = 0$, we need to multiply prefactor in (1.3) for taking into account the non-zero ground state energy of the related spin Calogero model [6, 13], and in the following we always suppose $m \neq 0$. A main purpose of the present Article is to introduce motifs as eigenstates of (1.1), and to give representation for the motifs. We can naturally define the supersymmetric RS polynomial from the partition function (1.3), and based on the recursion relation for those polynomials we can compute the degeneracy of motifs and the distribution function for the quasi-particles.

This Article is organized as follows. In § 2 we explain notations used in this paper. We review some important properties for the Schur polynomials following Refs. 14, 15. In § 3 we introduce a supersymmetric analogue of the Rogers–Szegö polynomial. This polynomial would reproduce the partition function (1.3) at some special values, and we study the recursion relation for these polynomials. In § 4 we introduce motif as eigenstates of the SP model, and we give the representation for motif using the supersymmetric skew Schur polynomials. In § 5 we calculate the distribution function for quasi-particles and the central charge by use of the recursion relation for the restricted partition function. We also discuss a relationship with the character formula in § 6. The last section is devoted to discussions and concluding remarks.

2 Preliminaries

2.1 Notation

We denote a q-polynomial as

$$(t; q)_N = \prod_{i=1}^{N} (1 - t\, q^{i-1}), \tag{2.1}$$

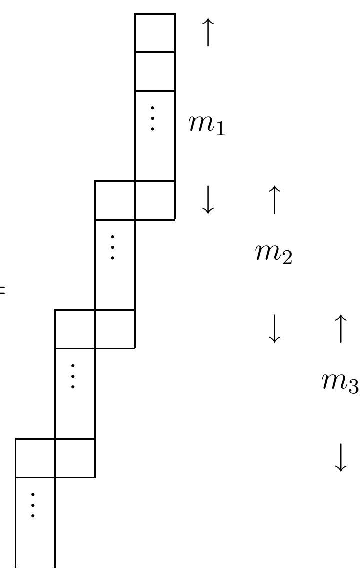
for $N > 0$, and we set $(t;q)_0 = 1$. For this polynomial, we have an identity [12],

$$\frac{1}{(t;q)_\infty} = \sum_{N=0}^{\infty} \frac{t^N}{(q;q)_N}. \tag{2.2}$$

A partition λ (see Refs. 14, 15 for detail) is given by a sequence of weakly decreasing positive integers, and we often write it as $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_m]$. The conjugate diagram $\tilde{\lambda}$ is given by flipping a diagram λ over its main diagonal (from upper left to lower right).

A skew diagram λ/μ is obtained by removing a smaller Young diagram $\mu = [\mu_1, \mu_2, \ldots]$ from a larger one λ that contains it. Hereafter, we often use the skew Young diagrams $\langle m_1, m_2, \ldots, m_r\rangle$, which denote the border strip of r-columns such that the length of the i-th column is m_i;

$$\langle m_1, m_2, \ldots, m_r\rangle =$$



(2.3)

2.2 Schur Polynomial

For each partition λ, we define the su(m) Schur polynomial $s_\lambda(x) = s_\lambda(x_1, \ldots, x_m)$ by the Jacobi–Trudi formula [14, 15];

$$s_\lambda(x) = \frac{\det\left((x_j)^{\lambda_i + m - i}\right)_{1 \le i,j \le m}}{\det\left((x_j)^{m-i}\right)_{1 \le i,j \le m}}. \tag{2.4}$$

Note that $s_\lambda(x) = 0$ when $\mathrm{length}(\lambda) > m$. For our brevity, we define the elementary and complete symmetric polynomials $e_N(x)$ and $h_N(x)$ respectively as

$$e_N(x) = s_{[1^N]}(x), \qquad h_N(x) = s_{[N]}(x). \tag{2.5}$$

The polynomial $h_N(x)$ is the sum of all distinct monomials of degree N in the variables x, while $e_N(x)$ is the sum of all monomials $x_{i_1} \dots x_{i_\ell}$ for all strictly increasing sequences $1 \le i_1 < \cdots < i_\ell \le m$. We have the generating functions for polynomials $e_N(x)$ and $h_N(x)$ as

$$\prod_{i=1}^{m} (1 + t\, x_i) = \sum_{N=0}^{m} e_N(x)\, t^N, \tag{2.6}$$

$$\prod_{i=1}^{m} \frac{1}{1 - t\, x_i} = \sum_{N=0}^{\infty} h_N(x)\, t^N. \tag{2.7}$$

We note that the Schur polynomials $s_\lambda(x)$ can be also given as [15]

$$\begin{aligned} s_\lambda(x) &= \det \left(h_{\lambda_i + j - i}(x) \right)_{1 \le i,j \le \mathrm{length}(\lambda)} \\ &= \det \left(e_{\tilde{\lambda}_i + j - i}(x) \right)_{1 \le i,j \le \mathrm{length}(\tilde{\lambda})}. \end{aligned} \tag{2.8}$$

The above definitions for the Schur polynomials can be generalized straightforwardly to the skew Young diagram λ/μ. We define the skew Schur polynomials by [14, 15]

$$\begin{aligned} s_{\lambda/\mu}(x) &= \det \left(h_{\lambda_i - \mu_j + j - i}(x) \right)_{1 \le i,j \le \mathrm{length}(\lambda)} \\ &= \det \left(e_{\tilde{\lambda}_i - \tilde{\mu}_j + j - i}(x) \right)_{1 \le i,j \le \mathrm{length}(\tilde{\lambda})}. \end{aligned} \tag{2.9}$$

where $\tilde{\lambda}$ and $\tilde{\mu}$ denote the conjugate partitions.

2.3 Supersymmetric Schur Polynomial

The $\mathrm{su}(m|n)$ supersymmetric analogue of the Schur polynomials can be defined as follows. As an extension of (2.8), we define the supersymmetric Schur polynomials (sometimes called as the hook Schur polynomial, or bisymmetric polynomial) $S_\lambda(x, y) = S_\lambda(x_1, \dots, x_m, y_1, \dots, y_n)$ as (see, e.g., Ref. 15)

$$S_\lambda(x, y) = \det \left(c_{\lambda_i + j - i} \right)_{1 \le i,j \le \mathrm{length}(\lambda)}, \tag{2.10}$$

where $c_N = c_N(x, y)$ is given by

$$\frac{\prod_{j=1}^{n}(1 + t\, y_j)}{\prod_{i=1}^{m}(1 - t\, x_i)} = \sum_{N=0}^{\infty} c_N\, t^N. \tag{2.11}$$

The coefficients c_N correspond to a supersymmetric analogue of the complete symmetric polynomial $h_N(x)$ (2.7). One easily sees a duality;

$$S_\lambda(x_1, \ldots, x_m, y_1, \ldots, y_n) = S_{\tilde{\lambda}}(y_1, \ldots, y_n, x_1, \ldots, x_m),$$

where $\tilde{\lambda}$ is the conjugate partition.

We can also define the supersymmetric analogue of the elementary symmetric polynomials. For our later convention, we set the supersymmetric elementary polynomials as

$$E_N(x, y) = S_{[1^N]}(x, y). \tag{2.12}$$

Based on identities (2.6), (2.7), and (2.11), the supersymmetric elementary function is decomposed [16, 17] as

$$E_N(x, y) = \sum_{k=0}^{N} e_k(x) \cdot h_{N-k}(y).$$

By using this relation it is easy to see that $E_N(x, y) \neq 0$ for arbitrary N. We note that the generating function for the polynomials $E_N(x, y)$ is given by

$$\frac{\prod_{i=1}^{m}(1 - t\, x_i)}{\prod_{j=1}^{n}(1 + t\, y_j)} = \sum_{N=0}^{\infty} (-)^N\, E_N(x, y)\, t^N. \tag{2.13}$$

3 Recursion Relation for the Partition Function

As the Rogers–Szegö polynomial reproduces the partition function for the su(m) Polychronakos model at some special values [7], it is natural to study a supersymmetric analogue of

the RS polynomials which is related with the partition function (1.3). In this section we give the recursion relations for such supersymmetric RS polynomials in terms of the Young diagrams.

3.1 Supersymmetric Rogers–Szegö Polynomials

We define polynomials $H_N^{(m|n)}(x,y) = H_N^{(m|n)}(x_1,\dots,x_m,y_1,\dots,y_n;q)$ as

$$H_N^{(m|n)}(x,y) = \sum_{\substack{\sum_{i=1}^m a_i + \sum_{j=1}^n b_j = N \\ a_i \geq 0, \quad b_j \geq 0}} \frac{(q;q)_N}{\prod_{i=1}^m (q;q)_{a_i} \cdot \prod_{j=1}^n (q^{-1};q^{-1})_{b_j}} \cdot x_1^{a_1} \dots x_m^{a_m} \cdot \left(-\frac{y_1}{q}\right)^{b_1} \dots \left(-\frac{y_n}{q}\right)^{b_n}, \tag{3.1}$$

which we call the su$(m|n)$ supersymmetric Rogers–Szegö (SRS) polynomial. It is easy to see that the polynomial $H_N^{(m|n)}(x,y)$ gives the partition function (1.3) for the SP model (1.1),

$$\mathcal{Z}_N^{(m|n)}(q) = H_N^{(m|n)}(x=1,y=1). \tag{3.2}$$

Next we try to derive the recursion relation for the SRS polynomials (3.1). For this purpose, it is useful to give a generating function of the polynomial (3.1). So we introduce a function $G^{(m|n)}(t) = G^{(m|n)}(t;x_1,\dots,x_m,y_1,\dots,y_n;q)$ as

$$G^{(m|n)}(t) = \frac{1}{\prod_{i=1}^m (t\,x_i;q)_\infty \cdot \prod_{j=1}^n (-t\,y_j\,q^{-1};q^{-1})_\infty}. \tag{3.3}$$

We see that, by use of an identity (2.2), $G^{(m|n)}(t)$ is a generating function for the SRS polynomial (3.1);

$$G^{(m|n)}(t) = \sum_{N=0}^\infty \frac{H_N^{(m|n)}(x,y)}{(q;q)_N} \cdot t^N. \tag{3.4}$$

By definition (3.3), the function $G^{(m|n)}(t)$ satisfies q-difference equation,

$$\left(\prod_{j=1}^n (1+t\,y_j)\right) \cdot G^{(m|n)}(q\,t) = \left(\prod_{i=1}^m (1-t\,x_i)\right) \cdot G^{(m|n)}(t). \tag{3.5}$$

Substituting (3.4) to above equation and using the elementary symmetric polynomials (2.6), we see that the SRS polynomial satisfies the following recursion relation for any $(\max(n,m)+1)$-consecutive polynomials;

$$H_N^{(m|n)}(x,y) = \sum_{k=1}^{\max(n,m)} (-)^{k-1} \frac{(q;q)_{N-1}}{(q;q)_{N-k}} \cdot \big(e_k(x) - (-)^k\, q^{N-k}\, e_k(y)\big) \cdot H_{N-k}^{(m|n)}(x,y), \quad (3.6)$$

where we have set $H_k^{(m|n)}(x,y) = 0$ for $k < 0$. However, by using the expression (2.13) which contains supersymmetric elementary polynomials, we can rewrite the q-difference equation (3.5) as

$$G^{(m|n)}(q\,t) = \left(\sum_{N=0}^{\infty} (-)^N\, E_N(x,y)\, t^N \right) \cdot G^{(m|n)}(t).$$

Substituting (3.4) to above equation, we obtain another type of recursion relation for the SRS polynomials as

$$H_N^{(m|n)}(x,y) = \sum_{k=1}^{N} (-)^{k+1} \frac{(q;q)_{N-1}}{(q;q)_{N-k}} \cdot E_k(x,y) \cdot H_{N-k}^{(m|n)}(x,y). \quad (3.7)$$

Notice that, since the elementary polynomials $E_k(x,y)$ do not vanish for arbitrary k, the polynomials $H_N^{(m|n)}(x,y)$ now depend on every lower degree polynomials $H_j^{(m|n)}(x,y)$ for $j < N$ contrary to the first type recursion relation (3.6). In the case of $n = 0$, both (3.6) and (3.7) reduce to the recursion relation for the Rogers–Szegö polynomial [7, 11].

3.2 Supersymmetric Skew Schur Polynomial

We define another q-polynomials $F_N^{(m|n)}(x,y) = F_N^{(m|n)}(x_1,\ldots,x_m,y_1,\ldots,y_n;q)$ following Ref. 18;

$$F_N^{(m|n)}(x,y) = \sum_{r=1}^{N} \sum_{\substack{m_1+\cdots+m_r=N \\ m_r \geq 1}} q^{\frac{N(N+1)}{2} - \sum_{i=1}^{r} (m_1+\cdots+m_i)} \times S_{\langle m_1,\ldots,m_r\rangle}(x,y), \quad (3.8)$$

where $S_{\langle m_1,\ldots,m_r\rangle}(x,y)$ is the supersymmetric skew Schur polynomial for the skew Young diagram $\langle m_1,\ldots,m_r\rangle$ (2.3). Being motivated from the expression of the skew Schur polynomials (2.9) in non-supersymmetric case, we define the above mentioned supersymmetric

Schur polynomials through the supersymmetric elementary polynomials (2.12) as

$$S_{\langle m_1,\ldots,m_r\rangle}(x,y)=\begin{vmatrix} E_{m_r} & E_{m_r+m_{r-1}} & \cdots & \cdots & E_{m_r+\cdots+m_1} \\ 1 & E_{m_{r-1}} & E_{m_{r-1}+m_{r-2}} & \cdots & E_{m_{r-1}+\cdots+m_1} \\ 0 & 1 & E_{m_{r-2}} & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & E_{m_1} \end{vmatrix}. \tag{3.9}$$

Remark that, since $E_N(x,y)\neq 0$ for arbitrary N, $S_{\langle m_1,\ldots,m_r\rangle}(x,y)$ will be non-trivial for arbitrary set of $m_1,\ m_2,\ldots,m_r$. Expanding the determinant in (3.9) along the first row, we get a recursion relation for the supersymmetric Schur polynomials as

$$S_{\langle m_1,\ldots m_r\rangle}(x,y)=\sum_{i=1}^{r}(-)^{i+1}\,E_{m_r+\cdots+m_{r-i+1}}(x,y)\cdot S_{\langle m_1,\ldots,m_{r-i}\rangle}(x,y).$$

By substituting the above relation to the expression (3.8) and following Appendix A in Ref 18, we find that the polynomial $F_N^{(m|n)}(x,y)$ satisfies the recursion relation,

$$F_N^{(m|n)}(x,y)=\sum_{k=1}^{N}(-)^{k+1}\frac{(q;q)_{N-1}}{(q;q)_{N-k}}\cdot E_k(x,y)\cdot F_{N-k}^{(m|n)}(x,y), \tag{3.10}$$

which is exactly same with the recurrence relation for the polynomials $H_N^{(m|n)}(x,y)$ (3.7). We can easily recognize that the initial conditions are same for two sets of polynomials, $H_N^{(m|n)}(x,y)$ and $F_N^{(m|n)}(x,y)$, *e.g.*,

$$\begin{aligned}
&F_0^{(m|n)}(x,y)=1,\\
&F_1^{(m|n)}(x,y)=S_{[1]}(x,y),\\
&F_2^{(m|n)}(x,y)=q\,S_{[1^2]}(x,y)+S_{[2]}(x,y),\\
&H_0^{(m|n)}(x,y)=1,\\
&H_1^{(m|n)}(x,y)=\sum_i x_i+\sum_j y_j,\\
&H_2^{(m|n)}(x,y)=\sum_i {x_i}^2+q\,\sum_j {y_j}^2+(1+q)\Big(\sum_{i_1<i_2}x_{i_1}\,x_{i_2}+\sum_{i,j}x_i\,y_j+\sum_{j_1<j_2}y_{j_1}\,y_{j_2}\Big).
\end{aligned} \tag{3.11}$$

As a result, we conclude that

$$F_N^{(m|n)}(x,y)=H_N^{(m|n)}(x,y), \tag{3.12}$$

and that the polynomial $F_N^{(m|n)}(x=1, y=1)$ also gives the partition function $\mathcal{Z}^{(m|n)}(q)$ for the SP model.

4 Motif: Eigenstates of SP Model

We study eigenstates for the supersymmetric su$(m|n)$ Polychronakos model (1.1). Due to the Yangian symmetry of the system, we can use the motif [3, 10] which spans the Fock space of the Yangian invariant spin systems.

The motif d for $(N+1)$-site spin chain is given by an N sequence of 0's and 1's; $d = (d_1, d_2, \ldots, d_N)$ with $d_j \in \{0, 1\}$. A value $d_j = 1$ (resp. $d_j = 0$) denotes that the j-th energy level is occupied (resp. empty) by the quasi-particle. The energy of the SP model is given by

$$E(d) = \sum_{j=1}^{N} j \, d_j. \tag{4.1}$$

We remark that the energy of the Yangian invariant Haldane–Shastry type spin chain [1–3] is given by $E_{\rm HS} = \sum_j j\,(N-j)\, d_j$. We define $g_N(d)$ as a degeneracy of the motif $d = (d_1, \ldots, d_N)$ for $(N+1)$-site spin chain. We note that $g_N(d)$ generally depends on x and y, and that the degeneracy for each motif is given by setting $x_i = y_j = 1$. Then the partition function (1.3) is written as

$$\mathcal{Z}_{N+1}(q) = \sum_d q^{E(d)} \, g_N(d). \tag{4.2}$$

In the su(m) case, we have a selection rule for motifs such that m-consecutive 1's are forbidden. On the contrary, in the su$(m|n)$ supersymmetric case we do not have such selection rules; every sequence d is permitted generally [10,19]. From the first few polynomials (3.11) we can read as

$$\begin{aligned} g_0(\,) &= S_{[1]}(x,y) = e_1(x) + e_1(y), \\ g_1(0) &= S_{[2]}(x,y), \\ g_1(1) &= S_{[1^2]}(x,y). \end{aligned}$$

Below we give two methods to obtain the degeneracy for the general motifs.

4.1 Recursion Relation for Motif

We study the representation of motif based on the first recursion relation (3.6). Using (4.2), we can translate the recursion relation (3.6) for the SRS polynomials to that for motifs.

We give the several examples as follows. By putting some special m and n to the first recurrence relation (3.6) and translating it in terms of motif, we get the following equations;

(A). su(2);

$$\begin{aligned} g_N(d_1,\ldots,d_N) &= \delta_{d_N,0}\cdot e_1(x)\cdot g_{N-1}(d_1,\ldots,d_{N-1}) \\ &\quad - \left(\delta_{d_N,0}\cdot\delta_{d_{N-1},0} - \delta_{d_N,1}\cdot\delta_{d_{N-1},0}\right)\cdot e_2(x)\cdot g_{N-2}(d_1,\ldots,d_{N-2}). \end{aligned} \tag{4.3}$$

(B). su(2|1);

$$\begin{aligned} g_N(d_1,\ldots,d_N) &= (\delta_{d_N,0}\, e_1(x) + \delta_{d_N,1} e_1(y))\cdot g_{N-1}(d_1,\ldots,d_{N-1}) \\ &\quad - \left(\delta_{d_N,0}\delta_{d_{N-1},0} - \delta_{d_N,1}\delta_{d_{N-1},0}\right)\, e_2(x)\cdot g_{N-2}(d_1,\ldots,d_{N-2}). \end{aligned} \tag{4.4}$$

(C). su(2|2);

$$\begin{aligned} g_N(d_1,\ldots,d_N) &= (\delta_{d_N,0}\, e_1(x) + \delta_{d_N,1}\, e_1(y))\cdot g_{N-1}(d_1,\ldots,d_{N-1}) \\ &\quad - \Big(\left(\delta_{d_N,0}\,\delta_{d_{N-1},0} - \delta_{d_N,1}\,\delta_{d_{N-1},0}\right) e_2(x) \\ &\qquad - \left(\delta_{d_N,0}\,\delta_{d_{N-1},1} - \delta_{d_N,1}\,\delta_{d_{N-1},1}\right) e_2(y)\Big)\cdot g_{N-2}(d_1,\ldots,d_{N-2}). \end{aligned} \tag{4.5}$$

See that the recurrence equation for the su(2|2) case reduces to those for su(2|1) and su(2) cases by setting $e_2(y) = 0$ and $e_1(y) = e_2(y) = 0$, respectively.

By using these recurrence equations for the motifs, we can compute degeneracy for each motif by setting $x_i = y_j = 1$. We give some examples for $N = 4$ below.

(A). su(2) and su(1|1);

motif	energy	degeneracy for su(2)	degeneracy for su(1|1)
(000)	0	5	2
(100)	1	3	2
(010)	2	4	2
(001)	3	3	2
(101)	4	1	2
(110)	3	–	2
(011)	5	–	2
(111)	6	–	2
	total	2^4	2^4

(B). su(3) and su(2|1);

motif	energy	degeneracy for su(3)	degeneracy for su(2|1)
(000)	0	15	9
(100)	1	15	12
(010)	2	21	16
(001)	3	15	12
(101)	4	9	12
(110)	3	3	8
(011)	5	3	8
(111)	6	–	4
	total	3^4	3^4

4.2 Motif and Skew Young Diagram

We have shown that the representation of motif is given recursively by use of the first recursion relation (3.6). In this subsection, we shall show that, by use of the skew Young diagram and the polynomials $F_N^{(m|n)}(x,y)$, we can directly give the representation for motif. We can then give the decomposition rule for each motif, and clarify the hyper-multiplet structure in the spectrum of the SP model.

From a correspondence between a power of q in (3.8) and the energy for motif d in (4.2), we can define a map from motif d to the skew Young diagram $\langle m_1, \ldots, m_r \rangle$ [18]. A rule for translation is: we read a motif $d = (d_1, d_2, \ldots)$ from the left, and we add a box under (resp. left) the box when we encounter '$d_j = 1$' (resp. '$d_j = 0$'). One sees that there is a one-to-one correspondence between motifs and skew Young diagrams. As we have stressed

before, the supersymmetric skew Schur polynomials $S_{\langle m_1,\ldots,m_r\rangle}(x,y)$ defined in (3.9) do not vanish for arbitrary set of $m_1,\ldots,m_r$, and this fact proves that there is no selection rule for motif d while the m-consecutive 1's are forbidden in the non-supersymmetric su(m) case.

We give examples up to $N=4$ below. We have decomposed following a rule of the supersymmetric Young diagrams.

motif	skew Young diagram	decomposition
( )	$\langle 1\rangle$	$[1]$
(0)	$\langle 1,1\rangle$	$[2]$
(1)	$\langle 2\rangle$	$[1^2]$
(11)	$\langle 3\rangle$	$[1^3]$
(01)	$\langle 1,2\rangle$	$[2,1]$
(10)	$\langle 2,1\rangle$	$[2,1]$
(00)	$\langle 1,1,1\rangle$	$[3]$
(111)	$\langle 4\rangle$	$[1^4]$
(110)	$\langle 3,1\rangle$	$[2,1^2]$
(101)	$\langle 2,2\rangle$	$[2^2]\oplus[2,1^2]$
(011)	$\langle 1,3\rangle$	$[2,1^2]$
(100)	$\langle 2,1,1\rangle$	$[3,1]$
(010)	$\langle 1,2,1\rangle$	$[3,1]\oplus[2^2]$
(001)	$\langle 1,1,2\rangle$	$[3,1]$
(000)	$\langle 1,1,1,1\rangle$	$[4]$

5 Distribution Function

In the preceding sections, we have shown that the motifs, which are the eigenstates of the SP model, satisfy some recursion relation. As the motifs are composed of the "quasi-particles", we shall consider their distribution function. Once the recursion relation for the partition function is given, it is straightforward to compute the distribution function by use of the asymptotic form of the partition function. This method was originally introduced in Refs. 20, 21 for a study of the exclusion statistics [22]. Namely in the SP model the energy

dispersion relation is linear, so we can regard q^k as

$$V_k = \exp\left(-\beta(\varepsilon_k - \mu)\right), \tag{5.1}$$

in which β, μ, and ε_k denote the inverse of temperature, the chemical potential, and the k-th energy respectively. Correspondingly the polynomial $H_N^{(m|n)}(x, y)$ can be viewed as the *restricted* partition function φ_N, in which particles can occupy up to N-th energy level ε_N (here dispersion relation for energy ε_k is arbitrary). By assuming an asymptotic form of the restricted partition function

$$\varphi_k \sim w^{-k}, \tag{5.2}$$

we find that the occupation of state ε_k is given by

$$\langle n_{\mathrm{av}} \rangle \simeq \frac{1}{k} \cdot \frac{\partial}{\partial(\beta\,\mu)} \log \varphi_k = -\frac{\partial}{\partial(\beta\,\mu)} \log w, \tag{5.3}$$

where the spectral parameter $w = w(V)$ is a function of $V = \exp\left(-\beta\,(\varepsilon - \mu)\right)$, and it can be determined from the recursion relation for the restricted partition function. In the same way, we can compute the specific heat from the partition function, and we get the central charge c of the theory as [23–26]

$$c = -\frac{6}{\pi^2} \int_0^1 \mathrm{d}V\, \frac{1}{V} \log w(V), \tag{5.4}$$

which is often rewritten by use of the Rogers' dilogarithm function (see, *e.g.*, Ref. 27 and references therein). In this section we demonstrate for some simple supersymmetric cases.

(A). su(1|1);

The recursion relation for the su(1|1) SRS polynomials is read off as

$$H_N^{(1|1)}(x, y) = \left(x + q^{N-1}\, y\right)\, H_{N-1}^{(1|1)}(x, y),$$

which shows that the restricted partition function satisfies

$$\varphi_N = (1 + V_{N-1})\ \varphi_{N-1}. \tag{5.5}$$

We see using (5.2) that the spectral parameter is given by

$$w = (1 + V)^{-1}. \tag{5.6}$$

We thus obtain the central charge (5.4) as

$$c = \frac{1}{2},$$

which denotes that the theory coincides with the free fermion theory. This fact is simply realized by explicitly computing the character in § 6.

(B). su(m);

Based on a recursion relation for the RS polynomial, we see that the restricted partition function φ_k satisfies

$$\varphi_N = \sum_{i=1}^{m} (-)^{i-1} {}_m\mathrm{C}_i \, (1 - V_{N-1}) \, (1 - V_{N-2}) \, (1 - V_{N-i+1}) \, \varphi_{N-i}, \tag{5.7}$$

where ${}_m\mathrm{C}_i$ denote binomial coefficients. By assuming V_N does not change drastically for N, we find [21] that the spectral parameter w (5.2) satisfies the functional equation,

$$V = (1 - (1 - V) \, w)^m . \tag{5.8}$$

We also see that the central charge (5.4) is computed as

$$c = m - 1,$$

which is known as the central charge for the level-1 su(m) WZNW theory.

(C). su(2|1);

The recursion relation for the partition function for the su(2|1) SP model is given as

$$H_N^{(2|1)}(x, y) = \left(x_1 + x_2 + q^{N-1} \, y\right) \cdot H_{N-1}^{(2|1)}(x, y) - (1 - q^{N-1}) \cdot x_1 \, x_2 \cdot H_{N-2}^{(2|1)}(x, y),$$

which, in terms of the restricted partition function, is rewritten as

$$\varphi_N = (2 + V_{N-1}) \cdot \varphi_{N-1} - (1 - V_{N-1}) \cdot \varphi_{N-2}. \tag{5.9}$$

By using an asymptotic form (5.2), we see that the spectral parameter satisfies the functional equation,

$$(1-V)\,w^2-(2+V)\,w+1=0,$$

which gives

$$\langle n_{\rm av}^{\rm su(2|1)}\rangle \simeq -\frac{\partial}{\partial(\beta\,\mu)}\log\left(\frac{1}{2\,(1-V)}\left(2+V-\sqrt{V^2+8\,V}\right)\right). \tag{5.10}$$

We obtain the central charge from (5.4) as

$$c=\frac{3}{2}=1+\frac{1}{2},$$

which shows that the theory has one boson and one fermion.

(D). su(2|2);

We have the recursion relation for the restricted partition function as

$$\varphi_N=2\,(1+V_{N-1})\cdot\varphi_{N-1}-(1-V_{N-1})\,(1-V_{N-2})\cdot\varphi_{N-2}, \tag{5.11}$$

which gives the functional equation for the spectral parameter (5.2) as

$$(1-V)^2\,w^2-2\,(1+V)\,w+1=0,$$

The average number of particle and the central charge are respectively given as

$$\langle n_{\rm av}^{\rm su(2|2)}\rangle \simeq 2\,\frac{\partial}{\partial(\beta\,\mu)}\log\left(1+\sqrt{V}\right)=2\,\langle n^{\rm su(2)}\rangle, \tag{5.12}$$

$$c=2=1+2\times\frac{1}{2}.$$

This result is consistent with the fact that the theory includes one boson and two fermions.

(E). su(1|2);

The recursion relation for the restricted partition function is given by

$$\varphi_N=(1+2\,V_{N-1})\,\varphi_{N-1}+(1-V_{N-1})\,V_{N-2}\,\varphi_{N-2}. \tag{5.13}$$

Correspondingly we get the functional equation for the spectral parameter (5.2),

$$V\,(V-1)\,w^2 - (1+2\,V)\,w + 1 = 0.$$

One sees a duality: $w^{\mathrm{su}(1|2)}(V) = V^{-1}\cdot w^{\mathrm{su}(2|1)}(V^{-1})$. The distribution function (5.3) and the central charge (5.4) are respectively computed as follows;

$$\langle n_{\mathrm{av}}^{\mathrm{su}(1|2)}\rangle \simeq -\frac{\partial}{\partial(\beta\,\mu)}\log\left(\frac{1+2\,V-\sqrt{1+8\,V}}{2\,V\,(V-1)}\right), \tag{5.14}$$

$$c = 1 = 2\times\frac{1}{2},$$

which shows that the su(1|2) theory has two independent fermions. We also find a duality for the occupation of states as

$$\langle n_{\mathrm{av}}^{\mathrm{su}(1|2)}(V)\rangle + \langle n_{\mathrm{av}}^{\mathrm{su}(2|1)}(V^{-1})\rangle = 1. \tag{5.15}$$

In Fig. 1 we plot an average occupation of states at a temperature $T>0$ for su$(m|n)$ cases ($m\leq 2$, $n=0,1,2$) and for the free fermion case. We see that

$$\langle n_{\mathrm{av}}\rangle|_{\varepsilon\to-\infty} = \begin{cases} \frac{1}{2}, & \text{for su}(2), \\ 1, & \text{for others,} \end{cases} \qquad \langle n_{\mathrm{av}}\rangle|_{\varepsilon\to\infty} \simeq \begin{cases} \sqrt{V}, & \text{for su}(2|2), \\ \frac{1}{\sqrt{2}}\sqrt{V}, & \text{for su}(2|1), \\ 3\,V, & \text{for su}(1|2), \end{cases}$$

and that

$$\langle n_{\mathrm{av}}\rangle|_{\varepsilon=\mu} = \begin{cases} \frac{1}{2}, & \text{for su}(2|2), \\ \frac{4}{9}, & \text{for su}(2|1), \\ \frac{5}{9}, & \text{for su}(1|2). \end{cases}$$

To close this section, we observe that the central charge (5.4) for the su$(m|n)$ SRS polynomials ($m\neq 1$) will be given generally as

$$c = m - 1 + \frac{n}{2}. \tag{5.16}$$

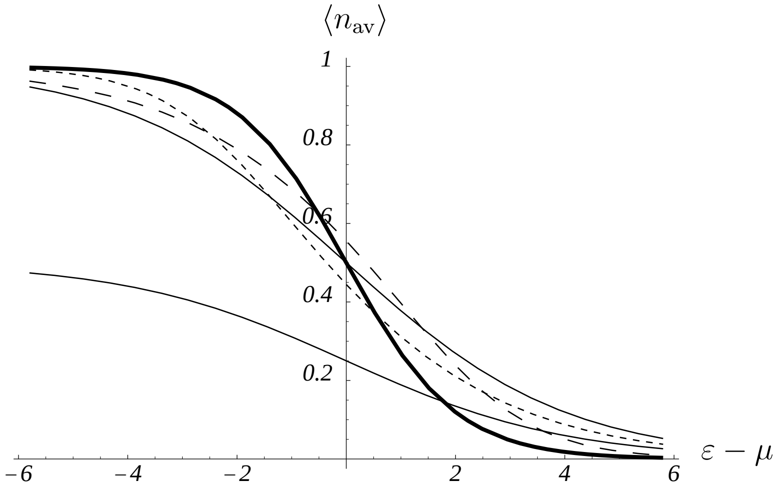


Figure 1: Average number of particle is plotted for free fermion and su$(m|n)$ cases $(m, n \leq 2)$. A bold line is the distribution function for the free fermion, $\langle n_{\rm av}\rangle = \dfrac{V}{1+V}$, and solid lines denote those for su(2) and su(2|2). A dotted and a dashed line respectively show the distribution function for su(2|1) and su(1|2) cases. The energy is given in units β^{-1}.

6 Character Formula

The SP model has the supersymmetric Yangian symmetry, and it is known that the first conserved operator for the Yangian operator is the Virasoro generator L_0. We can thus conclude that the character $\mathrm{ch}_\Lambda^{(m|n)}(q)$ for the su$(m|n)_1$ WZNW model with the highest weight Λ is related to the SRS polynomial as

$$\mathrm{ch}_\Lambda^{(m|n)}(q) = \lim_{\substack{N\to\infty \\ \text{condition}}} H_N^{(m|n)}(x=1, y=1; q). \tag{6.1}$$

We do not know how to treat the formula (6.1) for general cases, and below we give some examples.

(A). su(1|1);

It is easy to realize that the su(1|1) SRS polynomial is given as the mere q-product,

$$H_N^{(1|1)}(x, y; q) = \prod_{i=1}^{N} \left(x + q^{i-1}\, y\right). \tag{6.2}$$

We then obtain the character formula as

$$\begin{aligned}\mathrm{ch}_{\Lambda}^{(1|1)}(q) &= \lim_{N\to\infty} H_N^{(1|1)}(x=1, y=1; q) \\ &= \prod_{i=0}^{\infty} \left(1+q^i\right), \end{aligned} \tag{6.3}$$

which denotes that the su(1|1) theory coincides with the free spinless fermion theory as was demonstrated in Ref. 10.

(B). su(m) [7];

We have

$$\mathrm{ch}_{\Lambda_j}^{(m)}(q) = \lim_{\substack{N\to\infty \\ N\equiv j \mod m}} q^{\frac{m-1}{2\,m}N^2} \cdot H_N^{(m|0)}(x=1, y=1; q^{-1}), \tag{6.4}$$

which coincides with the character formula for su$(m)_1$ WZNW theory.

(C). su($1|n$);

Based on a numerical computation by use of Mathematica we suggest that

$$\begin{aligned}\mathrm{ch}_{\Lambda}^{(1|n)}(q) &= \lim_{N\to\infty} H_N^{(1|n)}(x_i=1, y=1; q) \\ &= \prod_{i=0}^{\infty} \left(1+q^i\right)^n. \end{aligned} \tag{6.5}$$

Indeed we can prove this identity from the definition (3.1) as follows [28];

$$\begin{aligned}\mathrm{ch}_{\Lambda}^{(1|n)}(q) &= \lim_{N\to\infty} \sum_{b_1,\ldots,b_n\geq 0} \frac{(q;q)_N}{(q;q)_{b_1}\ldots(q;q)_{b_n}\,(q;q)_{N-b_1-\cdots-b_n}}\, q^{\frac{1}{2}\sum_{j=1}^{n} b_j\,(b_j-1)} \\ &= \sum_{b_1,\ldots,b_n\geq 0} \frac{1}{(q;q)_{b_1}\ldots(q;q)_{b_n}}\, q^{\frac{1}{2}\sum_{j=1}^{n} b_j\,(b_j-1)} \\ &= \left(\sum_{b=0}^{\infty} \frac{1}{(q;q)_b}\, q^{\frac{1}{2}b\,(b-1)}\right)^n \\ &= \prod_{i=0}^{\infty}(1+q^i)^n. \end{aligned}$$

This identity also supports that the theory is written by n fermions (5.16).

7 Concluding Remarks

We have studied eigenstates of the supersymmetric extension of the Polychronakos spin chain, which is the Yangian invariant system. As it is known that the Yangian symmetry is realized in the level-1 WZNW theory [8, 9], the partition function can be regarded as the restricted character formula. We have introduced the supersymmetric Rogers–Szegö polynomials, and have shown that these polynomials give the partition function for the SP model. The SRS polynomials satisfy two types of recursion relations, (3.6) and (3.7); the first one is useful to find out the number of degenerate multiplets in each motif, while the second one gives us the decomposition rule or hyper-multiplet structure of each motif by the skew Young diagrams. The first recursion relation has another aspect. The spectrum (4.1) shows that the motif denote the quasi-particle excitation of the SP model. In fact the recursion relation denotes the exclusion statistics of the quasi-particles, and we have computed the distribution function following Ref. 21. One possible application of our results is for the edge states of the quantum Hall effect. As is well known from studies on the quantum Hall effect, the theory of the edge state of the Laughrin state is related to the Calogero–Sutherland model, and the quasi-particles have exclusion statistics. Our supersymmetric theories should also have close connections with the edge states. Especially the $c = 3/2$ su(2|1) theory gives quasi-hole excitations for the Pfaffian wave function [29], and the $c = 2$ su(2|2) theory is for the Haldane–Rezayi states [30]. We hope to discuss in detail in a future issue.

It has been found earlier that, the partition function of the vertex model, associated with the vector representation of $U_q(\hat{s\ell}_m)$ quantized affine algebra, exactly coincides with that of the $su(m)$ Polychronakos model [18]. So, as a future study, it might be interesting to investigate the partition function of the vertex model associated with the vector representation of $U_q(\hat{s\ell}_{m|n})$ algebra and examine whether such partition function also coincides with the polynomial partition function (3.1) of $su(m|n)$ SP model. Moreover, in analogy with the non-supersymmetric case [18], the spectral decomposition of the above mentioned vertex model should be intimately connected with the supersymmetric skew

Young diagrams and associated Schur functions which have been studied by us. Such intriguing relations between the vertex model with nearest-neighbor type interaction and Polychronakos spin chain with long-ranged interaction may enlighten our understanding of the common integrable structure behind these models. Another interesting problem might be to study the supersymmetric version of one-row Macdonald polynomial. As is well known, the one-row Macdonald polynomial can be viewed as a deformation of RS polynomial. Thus one can obtain a realization of one-row Macdonald polynomial by deforming the spinon representation associated with $su(m)$ RS polynomial [31]. In analogy with this non-supersymmetric case, it should be possible to find out a spinon representation for our supersymmetric RS polynomial (3.1). Furthermore, by deforming such spinon representation in an appropriate way, one may obtain some novel realization of supersymmetric one-row Macdonald polynomial.

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